Applying the flow equations to QCD

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Abstract

The effective $q\bar{q}$-interaction is derived from Lagrangian QCD in the front form by means of the flow equations. It coincides with previous results.

Before plunging into the paraphernalia of field theoretical details, Franz Wegner might allow me to outline his method [1] in yet an other language. Working with a Hamiltonian in an explicit representation, it is always possible to divide the complete Hilbert space arbitrarily into two pieces, the P- and the Q-space. Suppose, one is unable to solve the eigenvalue problem, because of say computer limitations. The method of flow equations allows then to unitarily transform the Hamiltonian matrix analytically into a block-diagonal effective Hamiltonian, *i.e.*

$$H = \begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix} \quad \longrightarrow \quad H_{\text{eff}} = \begin{pmatrix} PH_{\text{eff}}P & 0 \\ 0 & QH_{\text{eff}}Q \end{pmatrix}, \quad (1)$$

where P and $Q = 1 - P$ are the respective projection operators. (The methods also works for Hamiltonians with more than 2×2 blocks.) The 'reducible' matrix can then be diagonalized separately for either of the two spaces. The transformed Hamiltonian is a function of the continuous parameter $0 \le l \le \infty$,

$$\frac{d}{dl}H(l) = [\eta(l), H(l)], \qquad \eta(l) = [H_d(l), H(l)]. \qquad (2)$$

The generator of the transformation $\eta(l)$ is subject to some choice [3]. The Hamiltonian is separated conveniently into a block-diagonal part $H_d(l)$ and into the rest $H_r(l) \equiv H(l) - H_d(l)$ which is purely off-diagonal:

$$H_d(l) = \begin{pmatrix} PH(l)P & 0 \\ 0 & QH(l)Q \end{pmatrix}, \qquad H_r(l) = \begin{pmatrix} 0 & PH(l)Q \\ QH(l)P & 0 \end{pmatrix}. (3)$$

In many cases of practical interest one can interpret this rest as a 'residual interaction'. If the rest vanishes, or if it is exponentially small, one has

solved an important part of the problem. The generator is always off-diagonal: $P\eta(l)P = Q\eta(l)Q = 0$. The flow equations (2) for the diagonal and the off-diagonal block can then be divided into two coupled equations,

$$\frac{d}{dl}(PHP) = P\eta QHP - PHQ\eta P, \ \frac{d}{dl}(PHQ) = P\eta QHQ - PHP\eta Q, \ (4)$$

and into the trivial identity $P\eta Q = PHPHQ - PHQHQ$. The corresponding equation for QHQ will not be needed.

A possible measure for the 'block-off-diagonality' is $\mathcal{O} \equiv \mathrm{Tr} PHQHP \geq 0$. Its derivative is determined by the flow equations,

$$\frac{d}{dl}\mathcal{O} = \mathrm{Tr}\Big(P\eta Q(QHQHP - QHPHP)\Big) \hspace{2cm} (5)$$
$$+ \mathrm{Tr}\Big((PHPHQ - PHQHQ)Q\eta P\Big) = 2\mathrm{Tr}(P\eta Q\eta P) \leq 0,$$

because η is anti-hermitean. Since $\mathcal{O}$ is positive with a negative slope, the generator of Eq.(2) ensures a monotonic decrease: The off-diagonal block tends to vanish in the limit $l \rightarrow \infty$. With increasing l the effective Hamiltonian H_{eff} becomes 'more and more block-diagonal'.

Almost simultaneously with [1] and with a similar objective, Glazek and Wilson [2] have proposed the 'similarity transform' to render a Hamiltonian 'more and more diagonal'.

One should note the subtle but important difference between the two methods. Glazek and Wilson address to generate a band matrix whose width λ is decreased by analytical unitary transformations. In the asymptotic limit $\lambda \sim 1/l \rightarrow 0$ the Hamiltonian becomes diagonal. It is asked for too much, perhaps, to find a transform which diagonalizes a Hamiltonian completely. Flow equations have been applied to many problems, including solid-state physics, as well as field and gauge field theory. I do not even attempt to review all the work but refer to Wegner's talk [3]. I have taken great inspiration from his work with P. Lenz [4].

1   Applying the flow equations to quantum chromo-dynamics

In the next two sections I apply the flow equations to Non-Abelian gauge theory in the front form along the lines of earlier unpublished work [5]. The thinking in terms of block matrices is almost ideally suited for gauge theories, where the Fock-space expansions give quite naturally a block structure of the Hamiltonian, *cf.* for example Fig. 2 in [6]. It is thus reasonable to identify

the P-space with the lowest Fock-space sector, the one with one quark and one anti-quark $(q\bar{q})$. The Q-space is then literally the 'rest'. In front form, the Hamiltonian proper is $H = P^-$, with

$$P^- = T + V + F + S. \tag{6}$$

This as well as all other formulas in this section are taken without further notice from the Compendium [7], using the same notation and conventions as there. The *kinetic energy T* is the only piece of P^- which survives the limit $g \to 0$. The coupling constant g is related to the (strong) fine structure constant $\alpha_s = g^2/(4\pi\hbar c)$. The *vertex interaction V* is the relativistic interaction *per se*. It is linear in g and changes the particle number. The *instantaneous interactions F* and *S* (the 'forks' and the 'seagulls'), are quadratic in g and are artifacts of the light-cone gauge. Both may change the particle number. – As a technical trick and for gaining more transparency the instantaneous interactions are first omitted. They will be re-installed at the end of the calculation. By the same reason all color factors will be suppressed first and re-installed at the end. The Hamiltonian is then

$$H = T + V. \tag{7}$$

Since the front-form V has non-vanishing matrix elements between Fock states only, if the particle number differs by 1 and only 1, the initial P-space Hamiltonian is diagonal, $PHP = T$, see below. As opposed to that the effective Hamiltonian in the P-space has an effective interaction U,

$$PH_{\text{eff}}P = T + U. \tag{8}$$

The 'residual interaction' is then just the vertex interaction $H_r \equiv V$. The Q-space Hamiltonian remains a complicated operator for all l. It will not be addressed here. Objective of the present work is the calculation of U.

The way the flow equations work is that $V(l)$ de-creases while $U(l)$ in-creases for growing l. As emphasized by Wegner [3], flow equations generate a whole series of many-body interactions. They can be classified with the power of the coupling constant g. For the reason of demonstrating the method, I restrict here and below to the lowest non-trivial order in g without further mentioning.

The flow equations (4) are re-ordered and re-labeled as

$$\eta(l) = [T(l), V(l)], \quad \frac{dV(l)}{dl} = [\eta(l), T(l)], \quad \frac{dU(l)}{dl} = [\eta(l), V(l)]. \tag{9}$$

The relativistic kinetic energy is diagonal for all l:

$$T(l) = \int [d^3q] \left(\frac{\overline{m}^2(p;l) + \vec{p}_\perp^{\,2}}{p^+} \right)_q \left(b_q^\dagger b_q + d_q^\dagger d_q + a_q^\dagger a_q \right). \qquad (10)$$

The conventions in the Compendium include that $[d^3q]$ stands for the integration over the longitudinal momentum p^+ and the transversal momenta $\vec{p}_\perp$, as well as the summation over helicity, color (glue) and flavor. The single particle energies depend potentially on the flow parameter through the effective mass $\overline{m}^2(p;l)$. This dependence will be suppressed below, however, since it comes in higher order of g. Hence forward, p^+ and $p_\perp$ will be collected in the 3-vector $\vec{p} = (p^+, \vec{p}_\perp)$. The interaction term is

$$V(l) = \frac{1}{\sqrt{16\pi^3}} \int \frac{[d^3q_1]}{\sqrt{p_1^+}} \int \frac{[d^3q_2]}{\sqrt{p_2^+}} \int \frac{[d^3q_3]}{\sqrt{p_3^+}} \, \delta^{(3)}(\vec{p}_1 - \vec{p}_2 - \vec{p}_3) \, \overline{g}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l)$$

$$\times \left[b_1^\dagger b_2 a_3 \left(\overline{u}_1 \slashed{\epsilon}_3 u_2 \right) - d_1^\dagger d_2 a_3 \left(\overline{v}_2 \slashed{\epsilon}_3 v_1 \right) + a_1^\dagger d_2 b_3 \left(\overline{v}_2 \slashed{\epsilon}_1^\star u_3 \right) \right] + h.c.. \qquad (11)$$

The three-gluon vertex is omitted here because it does not contribute to the effective interaction to the lowest non-trivial order. The effective coupling $\overline{g}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l)$, with the initial value $\overline{g}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l = 0) = g$, contains all the l-dependence. The generator $\eta = [T, V]$ becomes

$$\eta(l) = \frac{1}{\sqrt{16\pi^3}} \int \frac{[d^3q_1]}{\sqrt{p_1^+}} \int \frac{[d^3q_2]}{\sqrt{p_2^+}} \int \frac{[d^3q_3]}{\sqrt{p_3^+}} \, \delta^{(3)}(\vec{p}_1 - \vec{p}_2 - \vec{p}_3) \, \overline{\eta}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l)$$

$$\times \left(b_1^\dagger b_2 a_3 \left[\overline{u}_1 \slashed{\epsilon}_3 u_2 \right] - d_1^\dagger d_2 a_3 \left[\overline{v}_2 \slashed{\epsilon}_3 v_1 \right] + a_1^\dagger d_2 b_3 \left[\overline{v}_2 \slashed{\epsilon}_1^\star u_3 \right] \right) - h.c.. \qquad (12)$$

The structure of η is very similar to V, because T is diagonal, thus

$$\overline{\eta}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l) = \overline{g}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l) c(\vec{p}_1, \vec{p}_2, \vec{p}_3; l). \qquad (13)$$

The commutator with T generates the 'commutator function'

$$c(\vec{p}_1, \vec{p}_2, \vec{p}_3; l) \equiv e(\vec{p}_1; l) - e(\vec{p}_2; l) - e(\vec{p}_3; l), \qquad (14)$$

with the single particle energies $e(\vec{p}; l) = (\overline{m}^2(p;l) + \vec{p}_\perp^{\,2})/p^+$. Symmetries like $\overline{g}(p_1', p_1, p_1' - p_1; l) = \overline{g}(p_1, p_1', p_1 - p_1'; l)$ are so obvious that they, as well as $\overline{\eta}(p_1', p_1, p_1' - p_1; l) = -\overline{\eta}(p_1, p_1', p_1 - p_1'; l)$, will not be mentioned hence forward. Similarly, the derivative $dV(l)/dl = [\eta(l), T(l)]$ becomes

$$\frac{dV(l)}{dl} = -\frac{1}{\sqrt{16\pi^3}} \int \frac{[d^3q_1]}{\sqrt{p_1^+}} \int \frac{[d^3q_2]}{\sqrt{p_2^+}} \int \frac{[d^3q_3]}{\sqrt{p_2^+}} \, \delta^{(3)}(\vec{p}_1 - \vec{p}_2 - \vec{p}_3) \qquad (15)$$

4

$$\times\, \overline{\eta}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l) c(\vec{p}_1, \vec{p}_2, \vec{p}_3; l)$$
$$\times\, \left(b_1^\dagger b_2 a_3 \left[\overline{u}_1 \not{\epsilon}_3 u_2 \right] - d_1^\dagger d_2 a_3 \left[\overline{v}_2 \not{\epsilon}_3 v_1 \right] + a_1^\dagger d_2 b_3 \left[\overline{v}_2 \not{\epsilon}_1^\star u_3 \right] \right) + h.c..$$

Finally, the derivative $dU(l)/dl = [\eta(l), V(l)]$ becomes

$$\frac{dU(l)}{dl} = -\frac{1}{16\pi^3} \int \frac{[d^3 q_1]}{\sqrt{p_1^+}} \int \frac{[d^3 q_2]}{\sqrt{p_2^+}} \int \frac{[d^3 q_3]}{\sqrt{p_3^+}} \int \frac{[d^3 q_1']}{\sqrt{p_1'^+}} \int \frac{[d^3 q_2']}{\sqrt{p_2'^+}} \int \frac{[d^3 q_3']}{\sqrt{p_3'^+}}$$

$$\left(\delta^{(3)}(\vec{p}_1 - \vec{p}_1' - \vec{p}_3) \delta^{(3)}(\vec{p}_2' - \vec{p}_2 - \vec{p}_3') \left[b_1^\dagger b_{1'} a_3, a_{3'}^\dagger d_2^\dagger d_{2'} \right] [\overline{u}_1 \not{\epsilon}_3 u_{1'}][\overline{v}_{2'} \not{\epsilon}_{3'}^\star v_2] \right.$$

$$\times \left(\overline{\eta}(\vec{p}_1, \vec{p}_1', \vec{p}_3; l) \overline{g}(\vec{p}_2', \vec{p}_2, \vec{p}_3'; l) + \overline{\eta}(\vec{p}_2', \vec{p}_2, \vec{p}_3'; l) \overline{g}(\vec{p}_1, \vec{p}_1', \vec{p}_3; l) \right) +$$

$$\delta^{(3)}(\vec{p}_1' - \vec{p}_1 - \vec{p}_3) \delta^{(3)}(\vec{p}_2 - \vec{p}_2' - \vec{p}_3') \left[d_2^\dagger d_{2'} a_{3'}, a_3^\dagger b_1^\dagger b_{1'} \right] [\overline{u}_1 \not{\epsilon}_3^\star u_{1'}][\overline{v}_{2'} \not{\epsilon}_{3'} v_2]$$

$$\left. \times \left(\overline{\eta}(\vec{p}_1', \vec{p}_1, \vec{p}_3; l) \, \overline{g}(\vec{p}_2, \vec{p}_2', \vec{p}_3'; l) + \overline{\eta}(\vec{p}_2, \vec{p}_2', \vec{p}_3'; l) \, \overline{g}(\vec{p}_1', \vec{p}_1, \vec{p}_3; l) \right) \right) \quad (16)$$

plus terms containing commutators like $[a^\dagger db, b^\dagger d^\dagger a]$ or $[a^\dagger db, a^\dagger a^\dagger a]$ contributing only in higher order of g. After execution of the various commutators it is obvious that U has the general form (*cf.* Eq.(8))

$$U(l) = \int [d^3 q_1] \int [d^3 q_2] \int [d^3 q_1'] \int [d^3 q_2'] \quad \delta^{(3)}(\vec{p}_1 - \vec{p}_1' + \vec{p}_2 - \vec{p}_2')$$
$$\times\, b_{q_1}^\dagger b_{q_1'} d_{q_2}^\dagger d_{q_2'} \, \overline{U}(q_1, q_2; q_1', q_2'; l). \quad (17)$$

It suffices thus to calculate the derivative of a c-number

$$\frac{d\overline{U}}{dl} = -\frac{1}{16\pi^3} \left(\theta(p_1^+ - p_1'^+) + \theta(p_1'^+ - p_1^+) \right)$$
$$\times \left(\overline{\eta}(\vec{p}_1', \vec{p}_1, \vec{p}_1' - \vec{p}_1; l) \overline{g}(\vec{p}_2, \vec{p}_2', \vec{p}_2 - \vec{p}_2'; l) \right.$$
$$\left. +\, \overline{g}(\vec{p}_1', \vec{p}_1, \vec{p}_1' - \vec{p}_1; l) \overline{\eta}(\vec{p}_2, \vec{p}_2', \vec{p}_2 - \vec{p}_2'; l) \right)$$
$$\times \frac{[\overline{u}(p_1, \lambda_1) \gamma^\mu u(p_1', \lambda_1')]}{\sqrt{p_1^+} \quad \sqrt{p_1'^+}} \frac{d_{\mu\nu}(q)}{q^+} \frac{[\overline{v}(p_2', \lambda_2') \gamma^\nu v(p_2, \lambda_2)]}{\sqrt{p_2'^+} \quad \sqrt{p_2^+}}. \quad (18)$$

The polarization tensor $d_{\mu\nu}$ arises by the summing over the helicities, *i.e.* $d_{\mu\nu}(q) \equiv \sum_\lambda \epsilon_\mu(q, \lambda) \epsilon_\nu^*(q, \lambda)$. The 4-momentum of the photon is denoted by q^μ. The step function is $\theta(x) = 1$ for $x \geq 0$, thus $\theta(x) + \theta(-x) = 1$.

2   Integrating the flow equations and shaping the notation

Eq.(9) gives the first order flow equations in operator form. After evaluating the matrix elements they reduce to two coupled algebraic equations:

$$\frac{d}{dl}\overline{g}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l) = -c(\vec{p}_1,\vec{p}_2,\vec{p}_3;0)\ \overline{\eta}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l), \tag{19}$$

$$\overline{\eta}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l) = c(\vec{p}_1,\vec{p}_2,\vec{p}_3;0)\ \overline{g}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l). \tag{20}$$

They can be combined into a single one

$$\frac{d}{dl}\overline{g}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l) = -c^2(\vec{p}_1,\vec{p}_2,\vec{p}_3;0)\ \overline{g}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l), \tag{21}$$

and integrated with the possible solution

$$\overline{g}(\vec{p}_1,\vec{p}_2,\vec{p}_3;l) = g\,\exp\left(-l\ c^2(\vec{p}_1,\vec{p}_2,\vec{p}_3;0)\right). \tag{22}$$

Since the p's appear in combinations like $(\vec{p}_1',\vec{p}_1,\vec{p}_1'-\vec{p}_1;l)$ (see $f.e.$ Eq.(18)), I introduce conveniently the energy differences

$$w_q \equiv -c(\vec{p}_1',\vec{p}_1,\vec{p}_1'-\vec{p}_1;0) = (p_1'-p_1)^- - p_1'^- + p_1^-, \tag{23}$$

$$w_{\bar{q}} \equiv -c(\vec{p}_2,\vec{p}_2',\vec{p}_2-\vec{p}_2';0) = (p_2-p_2')^- - p_2^- + p_2'^-. \tag{24}$$

Typically, they occur along the quark (w_q) and the anti-quark line $(w_{\bar{q}})$. They are in simple relationship to both the 4-momentum of the exchanged photon

$$q_\mu = p_{1\mu}' - p_{1\mu} + \eta_\mu\frac{w_q}{2} = p_{2\mu} - p_{2\mu}' + \eta_\mu\frac{w_{\bar{q}}}{2}, \tag{25}$$

and to the Feynman 4-momentum transfers along the two lines

$$Q_q^2 \equiv -(p_1'-p_1)^2 = q^+ w_q, \qquad\qquad Q_{\bar{q}}^2 \equiv -(p_2-p_2')^2 = q^+ w_{\bar{q}}. \tag{26}$$

The latter need not be equal in a Hamiltonian approach. The null 4-vector η^μ is defined by $\eta^\mu = (0,\vec{0},2)$ and cannot be confused with the operator η. The Feynman-momentum transfer Q^2 is always positive and a more physical quantity than an energy difference. The w's will be substituted therefore by the Q's as long as no misunderstanding can arise. In fact, I shall use often the *mean-square momentum transfer*

$$Q^2 \equiv \frac{1}{2}(Q_q^2 + Q_{\bar{q}}^2) = \frac{q^+}{2}(w_q + w_{\bar{q}}), \tag{27}$$

and the *mean-square difference*

$$\delta Q^2 \equiv \frac{1}{2}(Q_q^2 - Q_{\bar{q}}^2) = \frac{q^+}{2}(w_q - w_{\bar{q}}). \tag{28}$$

Also the polarization tensor $d_{\mu\nu}$ in Eq.(18) can be simplified with these defini-tions. Since $d_{\mu\nu}(q) = -g_{\mu\nu} + (\eta_\mu q_\nu + \eta_\nu q_\mu)/q^+$ appears always in combinations with the spinors, the Dirac equation $(p_1 - p_1')_\mu \overline{u}(p_1)\gamma^\mu u(p_1') = 0$ can be used to write

$$q_\mu \overline{u}(p_1, \lambda_1)\gamma^\mu u(p_1', \lambda_1') = \frac{w_q}{2}\eta_\mu \overline{u}(p_1, \lambda_1)\gamma^\mu u(p_1', \lambda_1'). \tag{29}$$

In the context of Eq.(18) one can use thus $d_{\mu\nu}(q) = -g_{\mu\nu} + \eta_\mu \eta_\nu Q^2/(q^+)^2$.

Following Glazek and Wilson [2], I introduce a dimensionless similarity func-tion $f(l)$ (with boundary conditions $f(0) \equiv 1$ and $f(\infty) \equiv 0$) by

$$f(\vec{p}_1, \vec{p}_2, \vec{p}_3; l) \equiv \frac{1}{g}\overline{g}(\vec{p}_1, \vec{p}_2, \vec{p}_3; l). \tag{30}$$

Note that f describes the decay rate of the off-diagonal vertex interaction V. The rate of decay, however, is not a unique function, as will be seen below. In line with recent developments [3], I want to keep $f = f(w; l)$ as a general function, particularly since $\overline{\eta}$ can be rewritten as

$$\overline{\eta}(l) = -\frac{1}{w}\left(\frac{d\ln f(w; l)}{dl}\right)\overline{g}(l). \tag{31}$$

Finally, I define

$$N_q \equiv -\int\limits_0^\infty dl' \frac{df(w_q; l')}{dl'} f(w_{\bar{q}}; l'), \qquad N_{\bar{q}} \equiv 1 - N_q, \tag{32}$$

which are kind of dimensionless occupation numbers.

With these definitions I return to the problem of integrating Eq.(18). Its rel-evant part is

$$\overline{\eta}(\vec{p}_1', \vec{p}_1, \vec{p}_1' - \vec{p}_1; l)\overline{g}(\vec{p}_2, \vec{p}_2', \vec{p}_2 - \vec{p}_2'; l) + \overline{\eta}(\vec{p}_2, \vec{p}_2', \vec{q}; l)\overline{g}(\vec{p}_1', \vec{p}_1, \vec{q}; l)$$
$$= -g^2 \left(\frac{1}{w_q}\frac{df(w_q; l)}{dl}f(w_{\bar{q}}; l) + \frac{1}{w_{\bar{q}}}\frac{df(w_{\bar{q}}; l)}{dl}f(w_q; l)\right). \tag{33}$$

The l-integration leads to

$$\int\limits_0^\infty dl' \left(\overline{\eta}(\vec{p}_1', \vec{p}_1, \vec{p}_1' - \vec{p}_1; l')\overline{g}(\vec{p}_2, \vec{p}_2', \vec{p}_2 - \vec{p}_2'; l') + \overline{\eta}(\vec{p}_2, \vec{p}_2', \vec{q}; l')\overline{g}(\vec{p}_1', \vec{p}_1, \vec{q}; l')\right)$$

$$= g^2 \left(\frac{N_q}{w_q} + \frac{N_{\bar{q}}}{w_{\bar{q}}} \right) = g^2 q^+ \left(\frac{N_q}{Q_q^2} + \frac{N_{\bar{q}}}{Q_{\bar{q}}^2} \right). \tag{34}$$

It is natural to assume [3] that the similarity function $f(w; l)$ is a homogeneous function of its arguments $f(w; l) = f(w^\kappa l)$, with some exponent κ. Two types of similarity functions are considered:

$$\text{Gaussian} \quad \text{cut-off} \ (\kappa = 2): \ f(w; l) = \exp(-w^2 l), \ N_q = \frac{w_q^2}{w_q^2 + w_{\bar{q}}^2}, \tag{35}$$

$$\text{Exponential cut-off} \ (\kappa = 1): \ f(w; l) = \exp(-wl), \quad N_q = \frac{w_q}{w_q + w_{\bar{q}}}. \tag{36}$$

The first case corresponds to Eq.(22). The second shows that the effective interaction depends explicitly on the similarity function. The requirement of block diagonalizing the Hamiltonian determines the generator only up to a unitary transformation [3].

3   The effective $q\bar{q}$-interaction from Hamiltonian flow

Collecting terms, the effective interaction from Eq.(18) becomes

$$\overline{U} = g^2 \frac{d_{\mu\nu}(q)}{16\pi^3} \frac{[\overline{u}(p_1, \lambda_1)\gamma^\mu u(p_1', \lambda_1')]}{\sqrt{p_1^+}\ \sqrt{p_1'^+}} \frac{[\overline{v}(p_2', \lambda_2')\gamma^\nu v(p_2, \lambda_2))]}{\sqrt{p_2'^+}\ \sqrt{p_2^+}} \left(\frac{N_q}{Q_q^2} + \frac{N_{\bar{q}}}{Q_{\bar{q}}^2} \right).$$

It is easy to restore the color factors. They sit only in the vertex interactions and go through untouched by the flow equations, thus

$$\overline{U} = g^2 \frac{d_{\mu\nu}(q)}{16\pi^3} \frac{[\overline{u}(p_1, \lambda_1)T^a\gamma^\mu u(p_1', \lambda_1')]}{\sqrt{p_1^+}\ \sqrt{p_1'^+}} \frac{[\overline{v}(p_2', \lambda_2')T^a\gamma^\nu v(p_2, \lambda_2))]}{\sqrt{p_2'^+}\ \sqrt{p_2^+}} \left(\frac{N_q}{Q_q^2} + \frac{N_{\bar{q}}}{Q_{\bar{q}}^2} \right).$$

If the $d_{\mu\nu}(q)$ is substituted, this turns into

$$\overline{U} = -\frac{g^2}{16\pi^3} \frac{[\overline{u}(p_1, \lambda_1)T^a\gamma^\mu u(p_1', \lambda_1')]}{\sqrt{p_1^+}\ \sqrt{p_1'^+}} \frac{[\overline{v}(p_2', \lambda_2')T^a\gamma^\nu v(p_2, \lambda_2)]}{\sqrt{p_2'^+}\ \sqrt{p_2^+}}$$
$$\times \left(g_{\mu\nu} - \eta_\mu \eta_\nu \frac{Q^2}{(q^+)^2} \right) \left(\frac{N_q}{Q_q^2} + \frac{N_{\bar{q}}}{Q_{\bar{q}}^2} \right). \tag{37}$$

The result holds actually for the general non-Abelian theory $SU(n_c)$.

It is equally straightforward to re-install the instantaneous interaction:

$$\overline{U}_{\mathrm{i}} = \frac{-g^2}{16\pi^3} \frac{[\overline{u}(p_1,\lambda_1)T^a\gamma^\mu u(p_1',\lambda_1')]}{\sqrt{p_1^+}\quad\sqrt{p_1'^+}} \frac{[\overline{v}(p_2',\lambda_2')T^a\gamma^\nu v(p_2,\lambda_2)]}{\sqrt{p_2'^+}\quad\sqrt{p_2^+}} \frac{\eta_\mu\eta_\nu}{(q^+)^2}, \quad (38)$$

cf. diagram $S_{3,1}$ in Table 5 of [7]. To the order considered here it is independent of l. Adding it in by $\overline{U} \longrightarrow \overline{U} + \overline{U}_{\mathrm{i}}$ gives

$$\overline{U} = -\frac{g^2}{16\pi^3} \frac{[\overline{u}(p_1,\lambda_1)T^a\gamma^\mu u(p_1',\lambda_1')]}{\sqrt{p_1^+}\quad\sqrt{p_1'^+}} \frac{[\overline{v}(p_2',\lambda_2')T^a\gamma^\nu v(p_2,\lambda_2)]}{\sqrt{p_2'^+}\quad\sqrt{p_2^+}}$$
$$\times \left[g_{\mu\nu}\left(\frac{N_q}{Q_q^2}+\frac{N_{\bar{q}}}{Q_{\bar{q}}^2}\right) + \eta_\mu\eta_\nu \left(\frac{N_q}{Q_q^2}-\frac{N_{\bar{q}}}{Q_{\bar{q}}^2}\right)\frac{\delta Q^2}{(q^+)^2}\right]. \quad (39)$$

Contracting the Lorentz indices (with $\gamma^\mu\eta_\mu = \gamma^+$) gives

$$\overline{U} = -\frac{\alpha_s}{4\pi^2} \frac{[\overline{u}(p_1,\lambda_1)T^a\gamma^\mu u(p_1',\lambda_1')]\,[\overline{v}(p_2',\lambda_2')T^a\gamma_\mu v(p_2,\lambda_2)]}{\sqrt{p_1^+\,p_2^+\quad p_1'^+\,p_2'^+}}\left(\frac{N_q}{Q_q^2}+\frac{N_{\bar{q}}}{Q_{\bar{q}}^2}\right)$$
$$-\frac{\alpha_s}{4\pi^2} \frac{[\overline{u}(p_1,\lambda_1)T^a\gamma^+ u(p_1',\lambda_1')]\,[\overline{v}(p_2',\lambda_2')T^a\gamma^+ v(p_2,\lambda_2)]}{\sqrt{p_1^+\,p_2^+\quad p_1'^+\,p_2'^+}}$$
$$\times \frac{\delta Q^2}{(q^+)^2}\left(\frac{N_q}{Q_q^2}-\frac{N_{\bar{q}}}{Q_{\bar{q}}^2}\right). \quad (40)$$

One notes the non-integrable quadratic singularity $(q^+)^2$ appearing both in the dynamic amplitude Eq.(37) and in the instantaneous interaction Eq.(38). In the perturbative calculation of the $q\bar{q}$ scattering amplitude, these singularities cancel each other exactly, see for example Sect. 3D of [6]. Also in the present approach they tend to cancel, but a residual piece remains in Eq.(40). It is weighted with a coefficient δQ^2 which possibly can be considered small, but beyond that, the term is dangerous because it violates manifestly gauge invariance (reflected in the matrix elements $\langle\gamma^+\gamma^+\rangle$). The quadratic singularity carries however also weighting factors which depend on the similarity function, see Eqs.(35,36). Inserting them, I get for the exponential cut-off:

$$N_q = \frac{Q_q^2}{Q_q^2+Q_{\bar{q}}^2}: \qquad \left(\frac{N_q}{Q_q^2}+\frac{N_{\bar{q}}}{Q_{\bar{q}}^2}\right)=\frac{1}{Q^2}, \qquad (41)$$

$$\frac{\delta Q^2}{(q^+)^2}\left(\frac{N_q}{Q_q^2}-\frac{N_{\bar{q}}}{Q_{\bar{q}}^2}\right)=0, \qquad (42)$$

and for the Gaussian cut-off:

$$N_q = \frac{(Q_q^2)^2}{(Q_q^2)^2 + (Q_{\bar{q}}^2)^2} : \qquad \left(\frac{N_q}{Q_q^2} + \frac{N_{\bar{q}}}{Q_{\bar{q}}^2} \right) = \frac{Q^2}{(Q^2)^2 + (\delta Q^2)^2}, \qquad (43)$$

$$\frac{\delta Q^2}{(q^+)^2} \left(\frac{N_q}{Q_q^2} - \frac{N_{\bar{q}}}{Q_{\bar{q}}^2} \right) = \frac{\delta Q^2}{(Q^2)^2 + (\delta Q^2)^2} \frac{\delta Q^2}{(q^+)^2} . \qquad (44)$$

For $\delta Q^2 \ll Q^2$ the Lorentz contracted part is obviously the same for both cut-offs. The potentially dangerous coefficient of $\langle \gamma^+ \gamma^+ \rangle$, however, behaves drastically different: It vanishes strictly for the exponential cut-off. For the Gaussian cut-off it can be small, at least it behaves now like $(\delta Q^2)^2$, but its precise behaviour near the important region of the Coulomb singularity at $Q^2 \sim 0$ must be left to future analysis. To easen my life I will work henceforward with the exponential cut-off and its gauge-invariant result.

Thus far, I have studied the Hamiltonian proper $H = P^-$. But it is often advantageous to work with the Lorentz-invariant 'light-cone Hamiltonian' $H_{\mathrm{LC}} = P_\mu P^\mu$ [6]. Its eigenvalues have the dimension of an invariant mass-squared. In the intrinsic frame ($\vec{P}_\perp = 0$) one has $H_{\mathrm{LC}} = P^+ P^-$. The effective interaction of H_{LC}, is obtained simply by $\overline{U} \Longrightarrow \overline{U} P^+$, thus

$$\overline{U} = -\frac{C_c}{4\pi^2} \frac{[\overline{u}(p_1, \lambda_1)\gamma^\mu u(p_1', \lambda_1')][\overline{v}(p_2', \lambda_2')\gamma_\mu v(p_2, \lambda_2)]}{\sqrt{x(1-x)x'(1-x')}} \frac{\alpha_s}{Q^2} \frac{1}{P^+}, \qquad (45)$$

where $x = p_1^+/P^+$ is the longitudinal momentum fraction of the quark. The color factor is $C_c = \sum_a T_{c_1, c_1'}^a T_{c_2', c_2}^a$. This finishes my objective: the so obtained effective interaction for QCD coincides identically with the result of lowest order obtained with the method of iterated resolvents [8].

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